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CIRCUMFERENTIAL GAP IN A CIRCULAR WAVEGUIDE  
EXCITED BY A DOMINANT CIRCULAR-ELECTRIC WAVE



By

J.E. Storer and L. S. Sheingold

March 1, 1953

Technical Report No. 166

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Abstract

The scattering matrix of a 360 degree circumferential slot in an infinitely long circular waveguide with an incident dominant circular-electric wave is obtained by a variational principle. The theory which should be very good for small gaps is shown to be in good agreement with the obtained experimental results.

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I. Introduction

Consider a circumferential gap in an infinitely long, perfectly conducting circular waveguide as shown in Fig. 1. The gap can be considered to be a region of free-space coupling two semi-infinite circular waveguides. Assuming that the gap width is small compared to the guide wavelength, the basic problem is to determine the effect of the gap on an incident  $TE_{01}$  wave.

The energy in the  $TE_{01}$  wave incident on the gap is partially reflected and partially transmitted. Some of the energy is coupled through the gap to the free space surrounding the waveguide. Since the power relationships are the essential quantities of interest, the problem is formulated in terms of the scattering matrix of a waveguide junction.

Intuitively, the small gap should have little effect on the propagation of an incident circular-electric wave. The circular-electric modes have the unique property that the wall surface currents are purely transverse. These transverse currents do not tend to charge the metallic boundary of the circumferential gap; hence little energy is expected to radiate into space. On the other hand, all TM modes are characterized by solely longitudinal currents on the guide wall, and all TE modes other than the circular-electric type have longitudinal components of current. It can be concluded that the modal energy of all extraneous modes will be coupled strongly to the surrounding medium.

The problem is formulated in terms of an integral equation for the axial component of the magnetic field on the gap surface. A variational principle is applied and a parameter  $\Delta$  is introduced representing the stationary parameter in the variational formulation of the integral equation. For simplification, even and odd excitations are used and the elements of the scattering matrix are expressed in terms of the appropriate  $\Delta$ 's. The restriction imposed by the theory is that the gap width must be small compared with wavelength.

## 2. Formulation of the Problem.

Since the following analysis pertains to circularly symmetrical waveguide modes, the electromagnetic field components can be derived from a scalar function which is independent of the  $\phi$ -coordinate. Assuming a time dependence  $\exp -i\omega t$ , the circular-electric modes are obtained by solving the equation

$$L E_{\phi}(r, z) = 0 \quad (1)$$

where the operator  $L$  acting on the  $\phi$ -components of the field is given by

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2} + k^2 \quad (2)$$

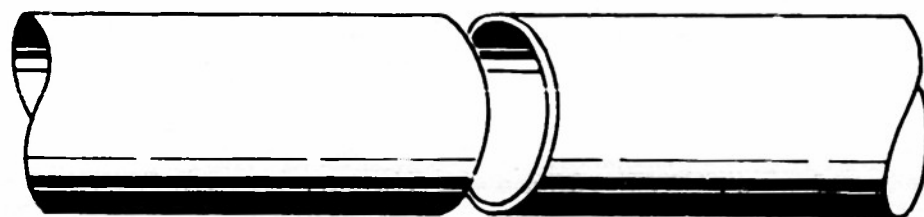


FIG. 1 A CIRCUMFERENTIAL GAP IN AN INFINITELY LONG CIRCULAR WAVEGUIDE

The other field components can be expressed in terms of  $E_\rho$  by:

$$\begin{aligned} H_r(r, z) &= \frac{1}{\omega\mu} \frac{\partial}{\partial z} E_\rho(r, z) \\ H_z(r, z) &= \frac{-1}{\omega\mu} \frac{1}{r} \frac{\partial}{\partial r} r E_\rho(r, z) \\ E_r &= E_z = H_\rho = 0 \end{aligned} \quad (3)$$

The walls of the cylinder are assumed to be perfectly conducting; therefore, the boundary conditions at the wall require that the tangential components of the electric field vanish. Hence, the boundary condition is

$$E_\rho(r, z) \Big|_{r=a} = 0 \quad (4)$$

A circumferential gap, of width  $2g$ , centered at  $z = 0$  is introduced in the infinite guide as indicated in Fig. 2. The region for  $r < a$  will be designated by superscript I and the region exterior to the guide,  $r > a$ , by superscript II.

In region II ( $r > a$ ) the field can be expressed in terms of a Green's function and the field distribution on the surface of the gap. The Green's function is defined by the equation

$$\begin{aligned} L G^{II}(r, r', z-z') &= - \frac{\delta(r-r')\delta(z-z')}{r'} \quad r > a \\ G^{II}(r, r', z-z') \Big|_{r=a} &= 0 \end{aligned} \quad (5)$$

By making use of Green's second theorem, an integral representation of the external field can be found in the form

$$E_\rho^{II}(r, z) = \int_{-g}^g \left[ \frac{1}{r'} \frac{\partial}{\partial r'} r' G^{II}(r, r', z-z') \right]_{r'=a} \xi(z') dz' \quad (6)$$

where

$$\xi(z') = \left[ E_\rho(r, z') \right]_{r=a}$$

While an explicit form for this Green's function can be obtained if desired, it is not necessary. Actually, it is simpler to determine the function occurring in the integral of (6). It is apparent from (6) that the integrand must satisfy the two conditions

$$L \left[ \frac{1}{r'} \frac{\partial}{\partial r'} r' G^{II}(r, r', z-z') \right]_{r'=a} = 0 \quad r > a \quad (7)$$

$$\left[ \frac{1}{r'} \frac{\partial}{\partial r'} r' G^{II}(r, r', z-z') \right]_{\substack{r'=a \\ r=a}} = \delta(z-z') \quad (8)$$

as well as the Sommerfeld radiation condition. By the use of the Fourier transform one obtains an expression of the form<sup>1</sup>

$$\left[ \frac{1}{r'} \frac{\partial}{\partial r'} r' G^{II}(r, r', z-z') \right]_{r'=a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\zeta(z-z')} \frac{H_1^{(1)}(\sqrt{k^2 - \zeta^2} r)}{H_1^{(1)}(\sqrt{k^2 - \zeta^2} a)} d\zeta.$$

where

$$\sqrt{k^2 - \zeta^2} = i\sqrt{\zeta^2 - k^2} \quad \text{for } \zeta > k \quad (9)$$

It is immediately apparent that such a choice satisfies (7) and (8) as well as the radiation conditions. Since the Green's function is unique, (9) is, therefore, the desired function.

A similar analysis can also be carried out for the region I ( $r > a$ ). Since there are fields propagating inside the pipe, the resulting expressions for the fields include an additional term representing the incident wave. In this case one obtains

$$E_{\phi}^I(r, z) = E_{\phi}^I \text{Inc.}(r, z) - \int_{-g}^g \left[ \frac{1}{r'} \frac{\partial}{\partial r'} r' G^I(r, r', z-z') \right]_{r'=a} \xi(z') dz' \quad (10)$$

Performing an analysis similar to that for region II, it is possible to show that

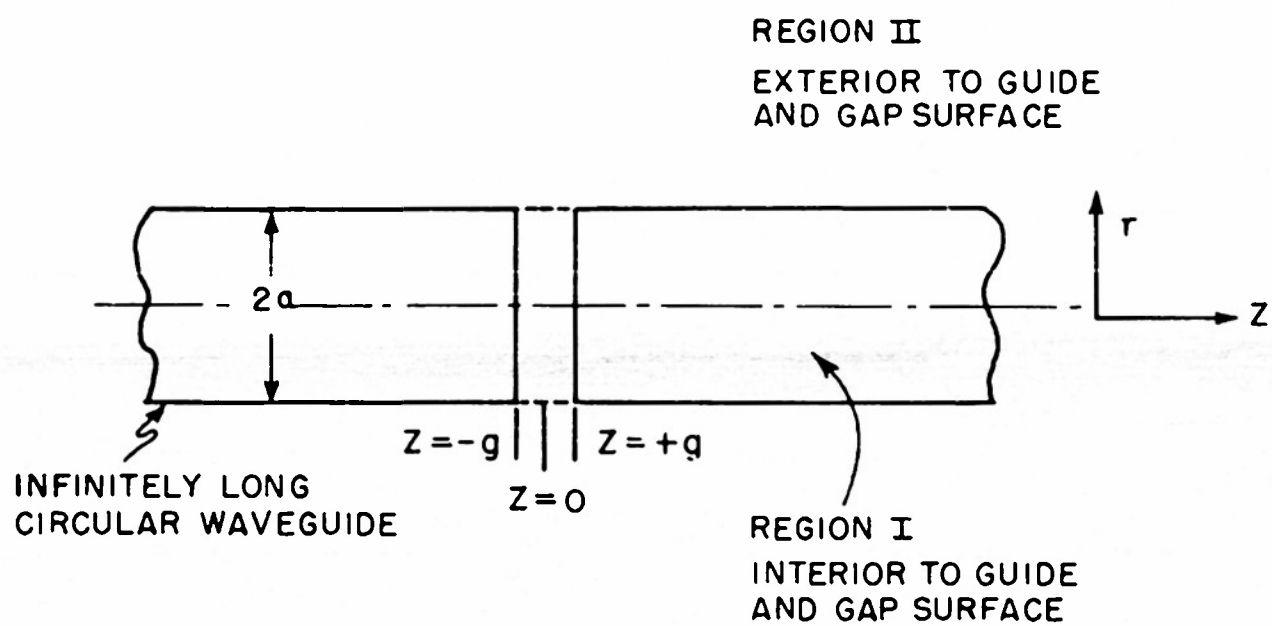


FIG. 2 ORIENTATION OF CYLINDRICAL COORDINATE SYSTEM



$$- \left[ \frac{1}{r} \frac{\partial}{\partial r} r' G^I(r, r', z-z') \right]_{r'=a} = \frac{1}{2\pi} \int_C e^{i\zeta(z-z')} \frac{J_1(\sqrt{k^2-\zeta^2} r)}{J_1(\sqrt{k^2-\zeta^2} a)} d\zeta \quad (11)$$

Assuming only a single mode propagates in the pipe, the Bessel function in the denominator of (11) vanishes at two places ( $\pm \gamma_1$ ) on the real axis, and at an infinite set of points ( $\pm i\gamma_n$ ) along the imaginary axis. The contour,  $C$ , used in (11) is that indicated in Fig. (3). The contour is chosen to pass around the singularities in the manner shown so that the Green's function represents only waves propagating away from the junction.

Thus, the entire electromagnetic field has been expressed in terms of the field on the surface of the gap. It remains now to determine integral equations for these fields. This can be done by matching the magnetic fields across the gap.

In particular, from (9) the magnetic field in the gap surface can be shown (except for dimensional constants) to be

$$H_z^{II}(a, z) \approx \left[ \frac{1}{r} \frac{\partial}{\partial r} r E_\phi^{II}(r, z) \right]_{r=a} = \int_{-g}^g K^{II}(z-z') \xi(z') dz' \quad (12)$$

where

$$K^{II}(z-z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\zeta(z-z')} H(\zeta) d\zeta \quad (13)$$

and

$$H(\zeta) = \frac{H_0^{(1)}(\sqrt{k^2-\zeta^2} a)}{H_1^{(1)}(\sqrt{k^2-\zeta^2} a)} \sqrt{k^2-\zeta^2} \quad (14)$$

Similarly, from (10)

$$H_z^I(a, z) \approx \left[ \frac{1}{r} \frac{\partial}{\partial r} r E_\phi^I(r, z) \right]_{r=a} = \left[ \frac{1}{r} \frac{\partial}{\partial r} r E_{\phi \text{Inc.}}^I(r, z) \right]_{r=a} +$$

$$+ \int_{-g}^g K^I(z-z') \xi(z') dz' \quad (15)$$

where

$$K^I(z-z') = \frac{1}{2\pi} \int_C e^{i\xi(z-z')} J(\xi) d\xi \quad (16)$$

and

$$J(\xi) = \frac{J_0(\sqrt{k^2 - \xi^2} a)}{J_1(\sqrt{k^2 - \xi^2} a)} \sqrt{k^2 - \xi^2} \quad (17)$$

Now

$$H_z^{II}(a, z) = H_z^I(a, z) \quad -g < z < g,$$

and, therefore, one obtains the integral equation

$$\left[ \frac{1}{r} \frac{\partial}{\partial r} r E_{\phi}^I(r, z) \right]_{r=a} = \int_{-g}^g \left[ K^{II}(z-z') - K^I(z-z') \right] \xi(z') dz' \quad -g < z < g \quad (18)$$

### 3. Variational Formulation.

It is now convenient to separate  $K^I(z-z')$  into two terms, as follows:

$$K^I(z-z') = \bar{K}^I(z-z') + \frac{1}{2\pi} \int_C e^{i\xi(z-z')} J(\xi) d\xi \quad (19)$$

where

$$\bar{K}^I(z-z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(z-z')} J(\xi) d\xi \quad (20)$$

i.e., integrals along the real axis and similar in form to those of (13). The remaining integrals along the contour,  $\bar{C}$ , indicated

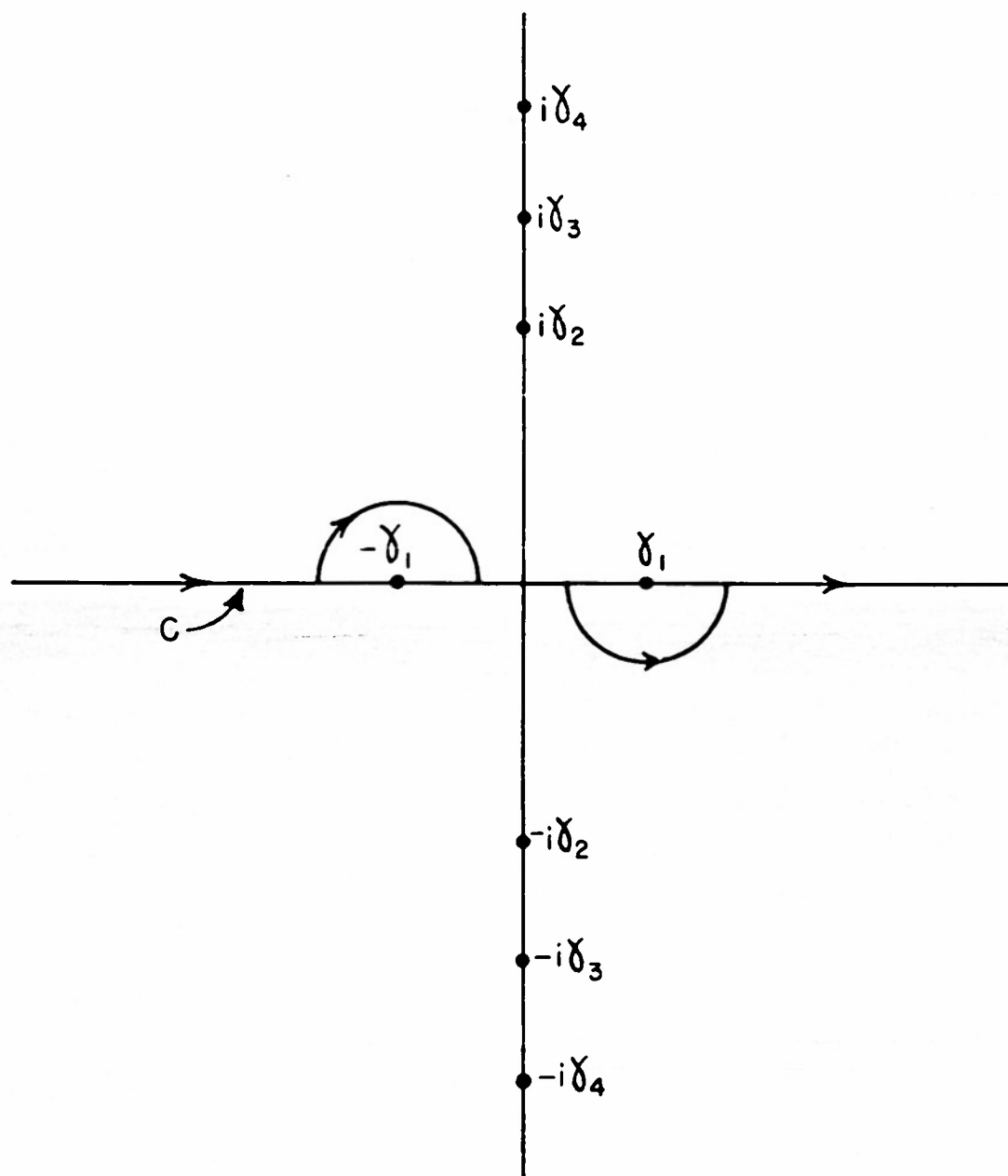


FIG. 3 CONTOUR C IN COMPLEX PLANE

in Fig. 4 can be evaluated by means of residues.

The integral equation then assumes the form

$$\begin{aligned} \frac{i\gamma_1 a^3}{\rho_1^2} \left[ \frac{1}{r} \frac{\partial}{\partial r} r E_{\rho \text{Inc.}}^I(r, z) \right]_{r=a} + \int_{-g}^g \cos \gamma_1(z-z') \xi(z') dz' \\ = \int_{-g}^g K(z-z') \xi(z') dz' \end{aligned} \quad (21)$$

where

$$K(z-z') = \frac{\gamma_1 a^3}{i\rho_1^2} \left[ K^{II}(z-z') - \bar{K}^I(z-z') \right] \quad (22)$$

and, the constants  $\rho_1$ ,  $\gamma_1$  are defined by

$$J_1(\rho_1) = 0 \quad ; \quad \gamma_1 = \sqrt{k^2 - \left(\frac{\rho_1}{a}\right)^2}$$

The fields incident in the waveguide can be expressed in the form

$$E_{\rho \text{Inc.}}^I(r, z) = \frac{\rho_1}{i\gamma_1 a^2} \frac{J_1\left(\rho_1 \frac{r}{a}\right)}{J_0(\rho_1)} \alpha(z) \quad (23)$$

Their  $z$ -dependence,  $\alpha(z)$ , is, in general, of the form

$$\alpha(z) = A e^{i\gamma_1 z} + B e^{-i\gamma_1 z}$$

i.e., fields incident on the gap from either the right or left. It is convenient to specialize this into even or odd incident fields, i.e.,

$$\alpha_e(z) = \cos \gamma_1 z \quad \text{even}$$

$$\alpha_o(z) = \sin \gamma_1 z \quad \text{odd}$$

For the even case,  $\xi_e(z) = \xi_e(-z)$  and hence,

$$\int_{-g}^g \cos \gamma_1(z-z') \xi_e(z') dz' = \alpha_e(z) \int_{-g}^g \alpha_e(z') \xi_e(z') dz'$$

Similarly, for the case of odd excitation  $\xi_o(z) = -\xi_o(-z)$  and hence,

$$\int_{-g}^g \cos \gamma_1(z-z') \xi_o(z') dz' = \alpha_o(z) \int_{-g}^g \alpha_o(z') \xi_o(z') dz'$$

Therefore, for either even or odd excitation, (23) can be inserted in the integral equation (21) which then becomes:

$$\alpha(z) \left[ 1 + \int_{-g}^g \alpha(z') \xi(z') dz' \right] = \int_{-g}^g K(z-z') \xi(z') dz' \quad (24)$$

Multiplying (24) by  $\xi(z)$ , integrating from  $-g$  to  $g$  and dividing by

$$\left[ \int_{-g}^g \alpha(z) \xi(z) dz \right]^2 \text{ yields the variational}$$

principle

$$\Delta = \frac{1 + \int_{-g}^g \alpha(z) \xi(z) dz}{\int_{-g}^g \alpha(z) \xi(z) dz} = \frac{\int_{-g}^g \int_{-g}^g K(z-z') \xi(z) \xi(z') dz dz'}{\left[ \int_{-g}^g \alpha(z) \xi(z) dz \right]^2} \quad (25)$$

(25) can be shown to have all the usual variational properties, i.e.,  $\Delta$  is stationary with respect to small changes of  $\xi(z)$  about its correct value and  $\Delta$  is independent of the overall amplitude

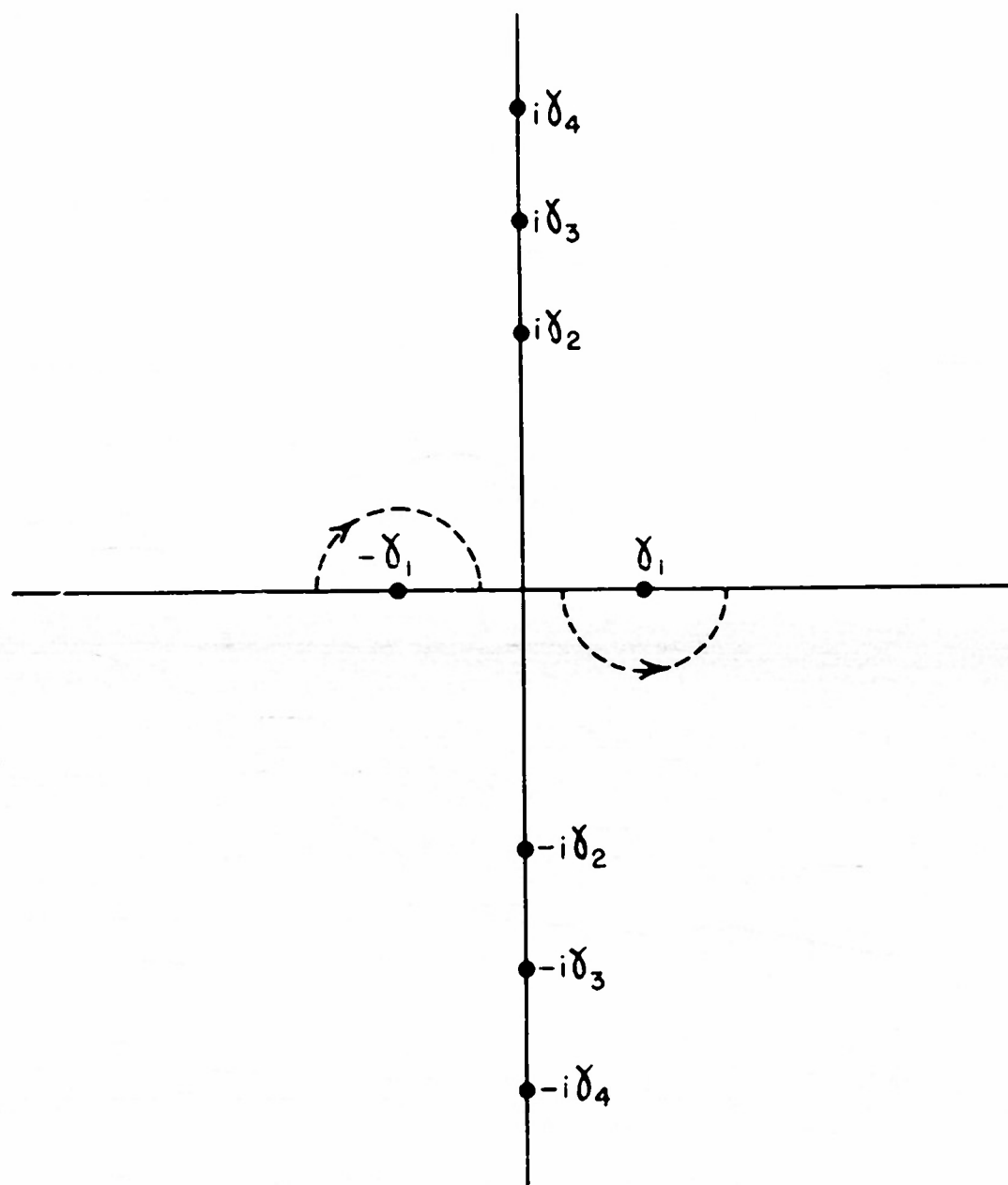


FIG. 4 CONTOUR  $\bar{C}$  IN COMPLEX PLANE

of  $\xi(z)$ .

Thus, it is possible to insert an approximate value for  $\xi(z)$  in (25) and obtain relatively more accurate values of  $\Delta$ . As will be seen later, the various  $\Delta$ 's can be related to the scattering matrix of the junction.

Referring to equations (13), (14), (17), (20), and (24), it is seen that  $K(z-z')$  can be expressed in the form

$$K(z-z') = \int_{-\infty}^{\infty} e^{i\xi(z-z')} \psi(\xi) d\xi \quad (26)$$

where, using the Wronskian for Bessel Functions,

$$\psi(\xi) = \frac{\gamma_1 a^2}{\pi^2 \rho_1^2} \frac{1}{J_1(\sqrt{k^2 - \xi^2} a) H_1^{(1)}(\sqrt{k^2 - \xi^2} a)} \quad (27)$$

The variational principles can then be written in the form

$$\Delta = \int_{-\infty}^{\infty} \Phi^2(\xi) \psi(\xi) d\xi \quad (28)$$

where

$$\begin{aligned} \Phi_e(\xi) &= \frac{\int_{-g}^g \cos \xi z \xi_e(z) dz}{\int_{-g}^g \cos \gamma_1 z \xi_e(z) dz} \\ \Phi_o(\xi) &= \frac{\int_{-g}^g \sin \xi z \xi_o(z) dz}{\int_{-g}^g \sin \gamma_1 z \xi_o(z) dz} \end{aligned} \quad (29)$$

It remains now to choose trial functions for  $\xi(z)$  and insert them into (28).

Consider the following choices

$$\bar{\xi}_e(z) = \sqrt{g^2 - z^2} \quad \bar{\xi}_o(z) = z \sqrt{g^2 - z^2} \quad (30)$$

It is apparent that these functions satisfy the symmetry conditions and have the correct static dependence on the edge of the gap. For small gaps they should, therefore, be good approximations to the correct fields. Inserting these into (29) yields

$$\Phi_e(\xi) = \frac{\gamma_1 J_1(\xi g)}{\xi J_1(\gamma_1 g)} \quad \Phi_o(\xi) = \frac{\frac{2}{(\xi g)^2} J_1(\xi g) - \frac{J_0(\xi g)}{\xi g}}{\frac{2 J_1(\gamma_1 g)}{(\gamma_1 g)^2} - \frac{J_0(\gamma_1 g)}{(\gamma_1 g)}} \quad (31)$$

Inserting (31) into (28), it can be shown (see Appendix) that

$$\Delta_e \doteq \frac{\gamma_1 a}{\rho_1^2} \left\{ \frac{(ka)^2}{4} + \frac{14}{\pi} \left(\frac{a}{g}\right)^2 \right\}$$

$$\Delta_o \doteq \frac{1}{\gamma_1 a \rho_1^2} \left\{ \frac{(ka)^4}{16} + \frac{132}{\pi} \left(\frac{a}{g}\right)^4 \right\}$$

It is to be noted that these results are only valid for small gaps, i.e.,  $g^2/\lambda^2 \ll 1$ . Results for larger gaps would be difficult to obtain theoretically because of the lack of satisfactory trial functions for  $\xi(z)$ . Moreover, it is virtually impossible to make satisfactory experimental measurements for large gaps since the interaction of the gap with the "free space" region outside the guide is of prime importance and a "free space" region is difficult to duplicate experimentally.



#### 4. Transmission-Line Representation

It is now desirable to express the scattering matrix elements in terms of  $\Delta_e$  and  $\Delta_o$ . This can be accomplished by examining the fields inside the waveguide at a terminal plane located a great distance from the gap discontinuity. The evaluation of the scattering (or impedance) matrix requires the knowledge of the explicit asymptotic expression of the Green's function of equation (11). This expression is obtained in the appendix. Substitution of the asymptotic form into equation (10) yields

$$E_{\rho}^I(r, z) = E_{\rho}^{I \text{ Inc.}}(r, z) + \frac{i\rho_1}{\gamma_1 a^2} \frac{J_1(\rho_1 \frac{r}{a})}{J_0(\rho_1)} e^{-i\gamma_1 z} \int_{-g}^g e^{i\gamma_1 z'} \xi(z') dz' \quad (33)$$

for the electric field in the far zone.

It then follows that the  $z$  dependence is of the form

$$E_{\rho}^I(r, z) \sim \alpha(z) + e^{-i\gamma_1 z} \int_{-g}^g e^{i\gamma_1 z'} \xi(z') dz' \quad (34)$$

Now for the even case ( $\alpha(z) = \cos \gamma_1 z$ ) this expression becomes

$$E_{\rho}^I(r, z) \sim e^{i\gamma_1 z} + \left[ 1 + 2 \int_{-g}^g \cos \gamma_1 z' \xi(z') dz' \right] e^{-i\gamma_1 z} \quad (35)$$

and the even reflection coefficient  $\Gamma_e$  can be defined as

$$\Gamma_e = 1 + 2 \int_{-g}^g \cos \gamma_1 z' \xi(z') dz' \quad (36)$$

Similarly, for odd excitation  $\alpha(z) = \sin \gamma_1 z$ , the odd reflection

coefficient  $\Gamma_o$  can be expressed as

$$\Gamma_o = -1 - 2 \int_g^o \sin \gamma_1 z' \xi(z') dz' \quad (37)$$

The integrals of equations (36) and (37) can be expressed in terms of  $\Delta$  from equation (25), resulting in

$$\Gamma_e = \frac{\Delta_e - 1}{\Delta_e + 1} \quad (38)$$

$$\Gamma_o = -\frac{\Delta_o - 1}{\Delta_o + 1}$$

The elements of the scattering matrix are then given by

$$\begin{aligned} s_{11} = s_{22} &= \frac{\Gamma_e + \Gamma_o}{2} = \frac{1}{1 + \Delta_o} - \frac{1}{1 + \Delta_e} \\ s_{12} &= \frac{\Gamma_e - \Gamma_o}{2} = 1 - \frac{1}{1 + \Delta_o} - \frac{1}{1 + \Delta_e} \end{aligned} \quad (39)$$

## 5. Experimental and Theoretical Results

Measurements of the characteristics of the circumferential gap were performed in a 5.755-in. i.d. brass tubing at a free-space wavelength of 10.000 cm. Deschamps' method was used to determine the elements of the scattering matrix of the circumferential gap. In comparing the theoretical and experimental results it was convenient to examine separately the absolute values and arguments of the scattering matrix elements. This method of presentation of data was chosen because the experimental values of amplitude and phase were determined separately by Deschamps' method.<sup>2</sup>

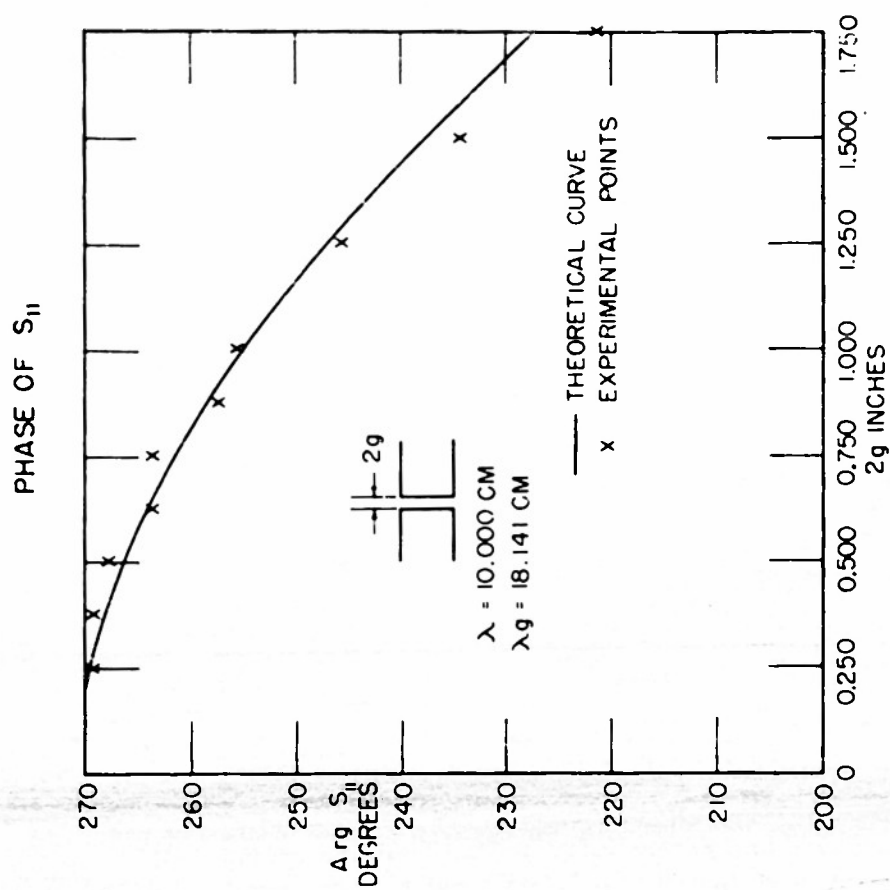
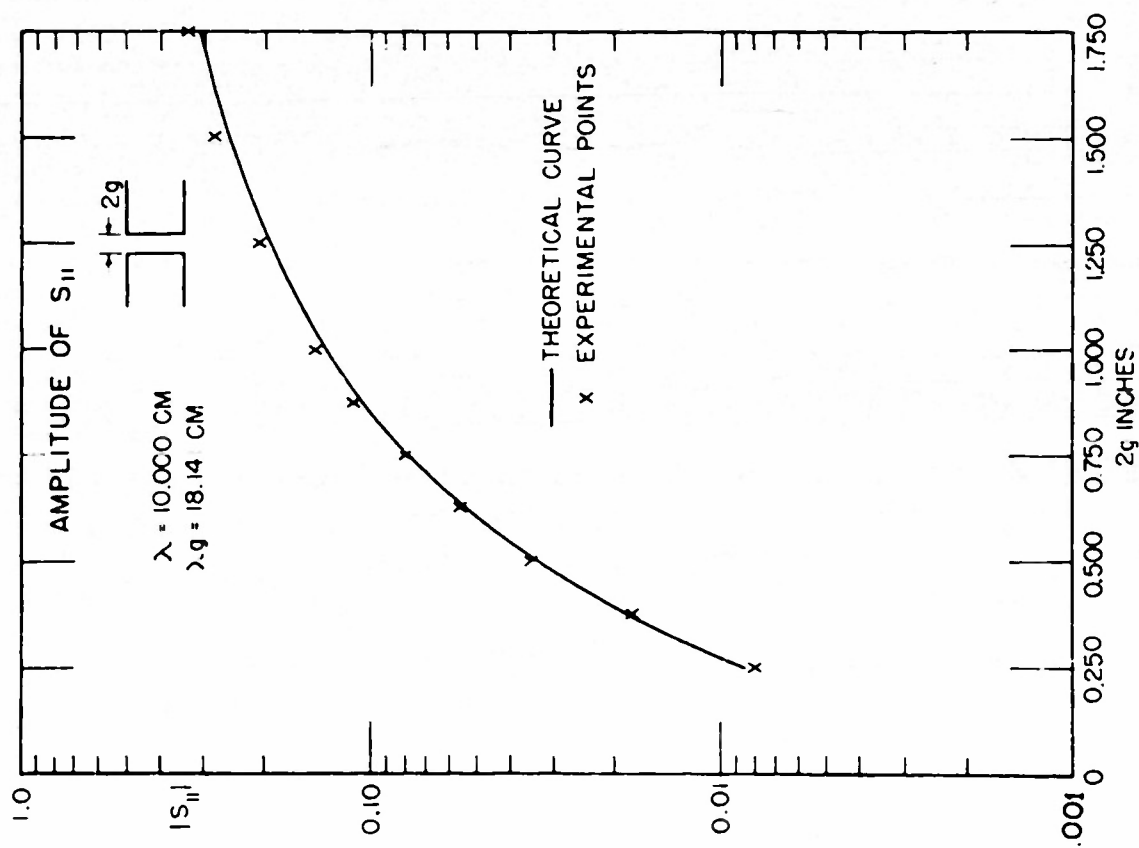


FIG. 5 MEASURED AND CALCULATED VALUES OF  $S_{11}$

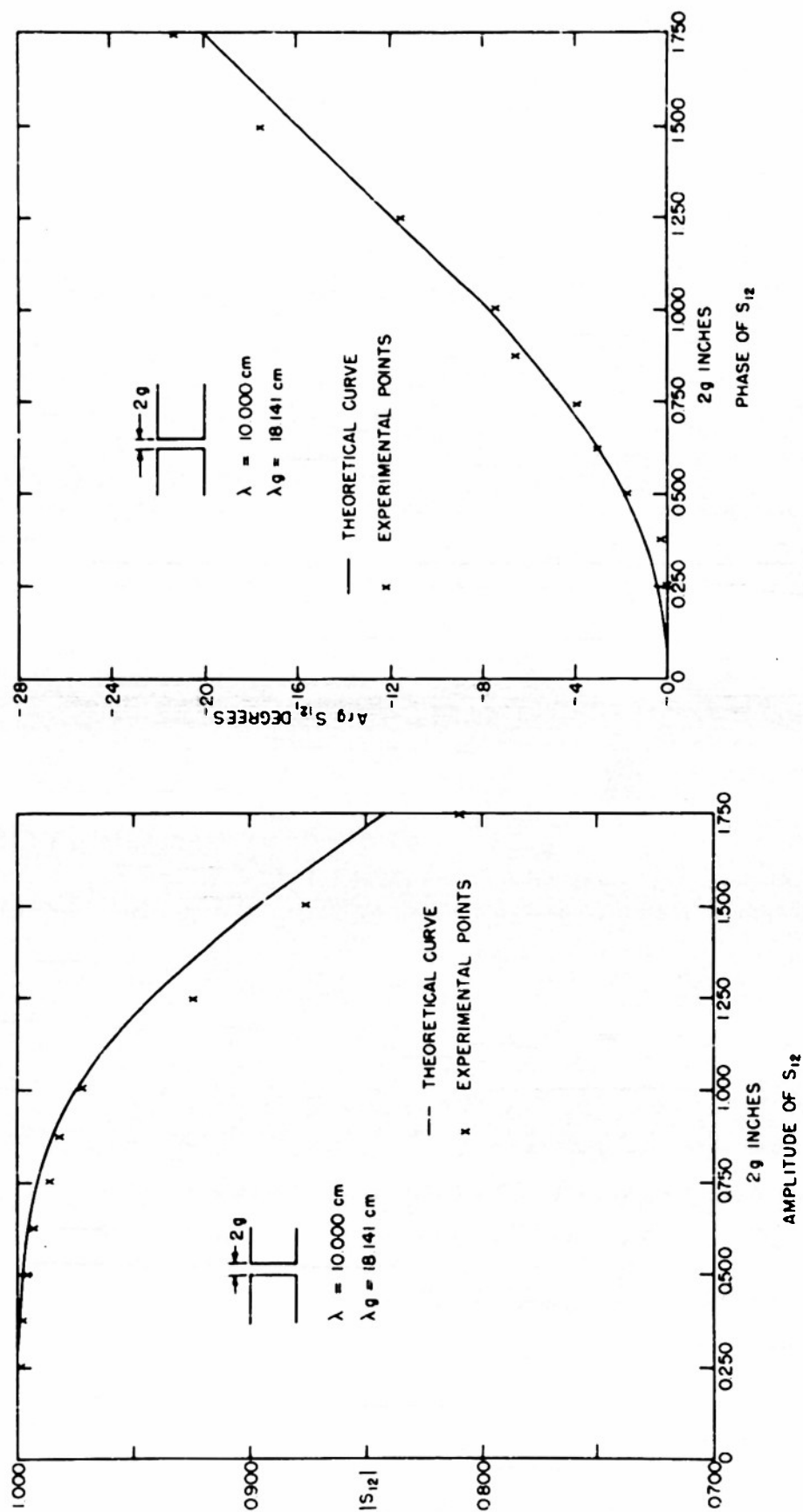


FIG. 6 MEASURED AND CALCULATED VALUES OF  $S_{12}$

The absolute values of  $S_{11}$ ,  $S_{22}$ , and  $S_{12}$  are then given by

$$|S_{11}| = |S_{22}| = \frac{|\Delta_e - \Delta_o|}{|1 + \Delta_o| |1 + \Delta_e|} \quad (40)$$

$$|S_{12}| = \frac{|\Delta_o \Delta_e - 1|}{|1 + \Delta_o| |1 + \Delta_e|}$$

and their arguments are found to be

$$\text{Arg } S_{11} = \text{Arg } S_{22} = \text{Arg}(\Delta_e - \Delta_o) - \text{Arg}(1 + \Delta_o) - \text{Arg}(1 + \Delta_e)$$

$$\text{Arg } S_{12} = \text{Arg}(\Delta_e \Delta_o - 1) - \text{Arg}(1 + \Delta_o) - \text{Arg}(1 + \Delta_e) \quad (41)$$

The experimental values are plotted on the same figure as the theoretically determined curves in Figs. 5 and 6.\* It is observed that the experimental value of  $|S_{11}|$  and  $|S_{12}|$  are in excellent agreement with the theory for values of  $2g$  up to 1.000 in. corresponding to  $\frac{g}{\lambda} = 0.127$ . It is recalled that the theory is only claimed to be valid for values of  $(\frac{g}{\lambda})^2 \ll 1$ . As seen from the phase characteristics of  $S_{11}$ , the experimental values lie slightly above the theoretical curve for small values of  $2g$ . It is observed in Fig. 6 that there is again excellent agreement between experiment and theory for values of  $\text{Arg } S_{12}$  such that  $2g \leq 1.000$ .

\* - - - - -  
A detailed description of the experimental techniques as well as a more complete discussion of the experimental results is given in Cruft Laboratory Technical Report No. 167. It should also be noted that in calculating the theoretical values of the scattering matrix elements from equation (32)  $-j$  is substituted for  $+i$  to conform to the convention established in measurement.

## Appendix A

Evaluation of the Variational Parameters  $\Delta_e$  and  $\Delta_o$ 

The determination of the elements of the scattering matrix of a circumferential gap requires the evaluation of

$$\Delta_e = \int_{-\infty}^{\infty} \Phi_e^2(\xi) \Psi(\xi) d\xi$$

$$\Delta_o = \int_{-\infty}^{\infty} \Phi_o^2(\xi) \Psi(\xi) d\xi$$
(A-1)

Since the assumed electric field distribution in the gap is a good approximation only for small values of  $\gamma_1 g$ , the parameters  $\Phi_e(\xi)$  and  $\Phi_o(\xi)$  can be expressed in the approximate form

$$\Phi_e(\xi) = \frac{\gamma_1 J_1(\xi g)}{\xi J_1(\gamma_1 g)} \approx \frac{2J_1(\xi g)}{\xi g} \quad \gamma_1 g \ll 1$$
(A-2)

$$\Phi_o(\xi) = \frac{\frac{2}{(\xi g)^2} J_1(\xi g) - \frac{J_o(\xi g)}{\xi g}}{\frac{2J_1(\gamma_1 g)}{(\gamma_1 g)^2} - \frac{J_o(\gamma_1 g)}{\gamma_1 g}} \approx \frac{8}{\gamma_1 g} \left\{ \frac{2}{(\xi g)^2} J_1(\xi g) - \frac{J_o(\xi g)}{\xi g} \right\}$$

$\gamma_1 g \ll 1$

and  $\Psi(\xi)$  is given by

$$\Psi(\xi) = \frac{\gamma_1 a^2}{\pi^2 \rho_1^2 J_1(\sqrt{k^2 - \xi^2} a) H_1^{(1)}(\sqrt{k^2 - \xi^2} a)}$$
(A-3)

It is convenient to determine separately the real and imaginary parts of  $\Delta_e$  and  $\Delta_o$ . Proceeding to evaluate R.P.  $\Delta_e$  one

observes that the integrand is real only for  $\xi < k$ . Therefore, assuming  $kg \ll 1$  as well as  $\gamma_1 g \ll 1$ , so that  $J_1(\xi g) \doteq \frac{\xi g}{2}$  for  $0 < \xi < k$ , it follows that

$$\begin{aligned} \text{R.P. } \Delta_e &= \frac{2\gamma_1 a^2}{\pi^2 \rho_1^2} \int_0^k \frac{d\xi}{J_1^2(\sqrt{k^2 - \xi^2} a) + \gamma_1^2 (\sqrt{k^2 - \xi^2} a)} \quad (\text{A-4}) \\ &= \frac{2\gamma_1 a^2}{\pi^2 \rho_1^2} \int_0^1 \frac{dt}{|H_1^{(1)}(\sqrt{1-t^2} ka)|^2} \end{aligned}$$

For  $t \ll 1$ , the denominator is large, and hence there is little contribution. On the other hand for  $t$  near zero, and with  $ka$  equal to 4.592, the Hankel function is already well approximated by its asymptotic form

$$|H_1^{(1)}(\sqrt{1-t^2} ka)|^2 \approx \frac{2}{\pi ka \sqrt{1-t^2}} \left[ 1 + \frac{3}{8(1-t^2)(ka)^2} \right] \quad (\text{A-5})$$

Inserting (A-5) into (A-4) results in

$$\text{R.P. } \Delta_e \doteq \frac{\gamma_1 k^2 a^3}{\pi \rho_1^2} \int_0^1 \frac{\sqrt{1-t^2}}{1 + \frac{\delta}{1-t^2}} dt \quad (\text{A-6})$$

where the quantity  $\delta$  is defined as

$$\delta = \frac{3}{8(ka)^2}$$

Equation (A-6) can be written as

$$\text{R.P. } \Delta_e \doteq \frac{\gamma_1 k^2 a^3}{4 \rho_1^2} \left[ 1 - \frac{4\delta}{\pi} \int_0^1 \frac{\sqrt{1-t^2}}{\delta + (1-t^2)} dt \right] \quad (\text{A-7})$$

Now since  $\delta \ll 1$ , it can be neglected in evaluating the remaining integral of (A-7). One then obtains

$$\text{R.P. } \Delta_e \doteq \frac{\gamma_1 k^2 a^3}{4\rho_1^2} [1 - 2\delta] \doteq \frac{\gamma_1 k^2 a^3}{4\rho_1^2} \quad (\text{A-8})$$

To evaluate the imaginary part of  $\Delta_e$ , examine the integral

$$\text{I.P. } \Delta_e = \text{I.P. } \frac{\gamma_1 a^2}{\pi^2 \rho_1^2} \int_{-\infty}^{\infty} \frac{\left[ \frac{2}{\xi g} J_1(\xi g) \right]^2}{J_1(\sqrt{k^2 - \xi^2} a) H_1^{(1)}(\sqrt{k^2 - \xi^2} a)} d\xi \quad (\text{A-9})$$

Noting that the integrand is imaginary for values of  $\xi > k$ , and that the major contribution to the integral occurs for  $\xi \gg k$ , one can write

$$\text{I.P. } \Delta_e = \frac{\gamma_1 a^3}{\pi \rho_1^2} \int_{-\infty}^{\infty} \xi \left[ \frac{2}{\xi g} J_1(\xi g) \right]^2 d\xi \quad (\text{A-10})$$

where the asymptotic expression

$$\begin{aligned} J_1(\sqrt{k^2 - \xi^2} a) H_1^{(1)}(\sqrt{k^2 - \xi^2} a) &\approx J_1(i\xi a) H_1^{(1)}(i\xi a) \\ &\approx \frac{1}{\pi i \xi a} \end{aligned} \quad (\text{A-11})$$

$$|\xi a| \gg 1$$

has been used. It then follows that\*

$$\text{I.P. } \Delta_e \doteq \frac{8\gamma_1 a}{\pi \rho_1^2} \left(\frac{a}{g}\right)^2 \int_0^{\infty} \frac{[J_1(\xi g)]^2}{\xi} d\xi =$$

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\*Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, London, 1945, p. 397.



$$\begin{aligned}
 &= \frac{8\gamma_1 a}{\pi \rho_1^2} \left(\frac{a}{g}\right)^2 \int_0^\infty \frac{[J_1(t)]^2}{t} dt \\
 &= \frac{4\gamma_1 a}{\pi \rho_1^2} \left(\frac{a}{g}\right)^2 \quad (A-12)
 \end{aligned}$$

In a similar fashion the real part of  $\Delta_0$  is given by

$$\text{R.P. } \Delta_0 = \text{R.P. } \frac{\gamma_1 a^2}{\pi^2 \rho_1^2 \gamma_1 g} \int_{-\infty}^{\infty} \frac{\left[ \frac{2}{(\xi g)^2} J_1(\xi g) - \frac{J_0(\xi g)}{\xi g} \right]^2}{J_1(\sqrt{k^2 - \xi^2} a) H_1^{(1)}(\sqrt{k^2 - \xi^2} a)} d\xi \quad (A-13)$$

It is seen again that the integrand has a real part when  $k < \xi$ . Again assuming  $kg \ll 1$ , and using the leading term in the expansion of the numerator, one obtains

$$\text{R.P. } \Delta_0 = \frac{2a^2}{\pi^2 \rho_1^2 \gamma_1} \int_0^k \frac{\xi^2}{|H_1^{(1)}(\sqrt{k^2 - \xi^2} a)|^2} d\xi \quad (A-14)$$

Again making the substitution  $t = \xi/k$ ,

$$\text{R.P. } \Delta_0 = \frac{2(ka)^3}{\pi^2 \rho_1^2 \gamma_1 a} \int_0^1 \frac{t^2 dt}{|H_1^{(1)}(\sqrt{1-t^2} ka)|^2} \quad (A-15)$$

Introducing the asymptotic expression  $|H_1^{(1)}(\sqrt{1-t^2} ka)|^2 \approx \frac{2}{\pi ka \sqrt{1-t^2}}$  this becomes

$$\begin{aligned}
 \text{R.P. } \Delta_0 &= \frac{(ka)^4}{\pi \rho_1^2 \gamma_1 a} \int_0^1 \sqrt{1-t^2} t^2 dt \\
 &= \frac{(ka)^4}{16 \rho_1^2 \gamma_1 a} \quad (\text{Peirce No. 145}) \quad (A-16)
 \end{aligned}$$

Finally, by using an argument similar to that which led from (A-9) to (A-10), the imaginary part of  $\Delta_o$  is seen to be of the form

$$\text{I.P. } \Delta_o = \frac{128}{\pi \rho_1^2 \gamma_1 a} \left(\frac{a}{g}\right)^4 I \quad (\text{A-17})$$

where

$$I = \int_0^\infty \left\{ \frac{4}{\xi^3 g^2} J_1^2(\xi g) - \frac{4}{\xi^2 g} J_1(\xi g) J_0(\xi g) + \frac{J_0^2(\xi g)}{\xi} \right\} d\xi \quad (\text{A-18})$$

By the substitution  $t = \xi g$ ,

$$I = \int_0^\infty \left\{ \frac{4}{t^3} J_1^2(t) - \frac{4}{t^2} J_1(t) J_0(t) + \frac{J_0^2(t)}{t} \right\} dt \quad (\text{A-19})$$

This integral can be expressed as

$$\begin{aligned} I &= \int_0^\infty \left\{ \frac{J_1^2(t)}{t} - \frac{d}{dt} \left[ \frac{J_1^2(t)}{t^2} + \frac{J_0(t) J_1(t)}{t} \right] \right\} dt \\ &= \int_0^\infty \frac{J_1^2(t)}{t} dt - \int_0^\infty \left[ \frac{J_1^2(t)}{t^2} + \frac{J_0(t) J_1(t)}{t} \right] dt = 1/4 \end{aligned} \quad (\text{A-20})$$

Therefore

$$\text{I.P. } \Delta_o = \frac{32}{\pi \rho_1^2 \gamma_1 a} \left(\frac{a}{g}\right)^4 \quad (\text{A-21})$$

Finally, the resultant expressions for the variational parameters are given by

$$\begin{aligned} \Delta_e &\doteq \frac{\gamma_1 a}{\rho_1^2} \left\{ \frac{(ka)^2}{4} + 1 \frac{4}{\pi} \left(\frac{a}{g}\right)^2 \right\} \\ \Delta_o &\doteq \frac{1}{\gamma_1 a \rho_1^2} \left\{ \frac{(ka)^4}{16} + 1 \frac{32}{\pi} \left(\frac{a}{g}\right)^4 \right\} \end{aligned} \quad (\text{A-22})$$

## Appendix B

Evaluation of the Explicit Asymptotic Expression of  
the Green's Function of Equation (11)

It is desired to derive an explicit form for the Green's function of equation (11) in order to evaluate the far-zone electric field in region I. The expression for the Green's function is given by

$$-\left[ \frac{1}{r'} \frac{\partial}{\partial r'} r' G^I(r, r', z-z') \right]_{r'=a} = \frac{1}{2\pi} \int_C e^{+i\xi(z-z')} \frac{J_1(\sqrt{k^2 - \xi^2} r)}{J_1(\sqrt{k^2 - \xi^2} a)} d\xi \quad (B-1)$$

where the contour C is chosen as in Fig. 3.

It is noted that the integrand is analytic except for simple poles at  $\xi = \pm \gamma_1$  and  $\xi = \pm i\gamma_n$  ( $n > 1$ ). Since the result is required for  $z \ll z'$  it is appropriate to close the contour with a large semi-circle in the lower half-plane. It is apparent that the integral along the semi-circle vanishes as its radius tends to infinity. Therefore, by the theory of residues, and noting that only the poles  $\xi = -\gamma_1$  and  $\xi = -i\gamma_n$  ( $n > 1$ ) are enclosed, one obtains

$$-\left[ \frac{1}{r'} \frac{\partial}{\partial r'} r' G^I(r, r', z-z') \right]_{r'=a} = \frac{2\pi i}{2\pi} \sum_{\substack{\xi = -\gamma_1 \\ \xi = -i\gamma_n \ (n>1)}} \text{Residue at} \quad (B-2)$$

Since  $e^{i\xi(z-z')}$  represents a rapidly attenuated function when  $z \ll z'$  and  $\xi$  is negative imaginary, all higher-order terms are negligible compared to the dominant-mode term.

Evaluating the residue corresponding to  $\xi = -\gamma_1$ , one obtains the desired expression

$$-\left[ \frac{1}{r'} \frac{\partial}{\partial r'} r' G^I(r, r', z-z') \right]_{r'=a} = \frac{i\rho_1 J_1(\rho_1 \frac{r}{a})}{\gamma_1 a^2 J_0(\rho_1)} e^{-i\gamma_1(z-z')} \quad (B-3)$$

$z \ll z'$

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